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1993 J. Phys. A: Math. Gen. 26 L1149

(http://iopscience.iop.org/0305-4470/26/22/004)

J. Phys. A: Math. Gen. 26 (1993) L1149–L1156. Printed in the UK

LETTER TO THE EDITOR

On toy ageing

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Received 10 August 1993

Abstract. We consider the dynamics of a simple one-dimensional model and we discuss the phenomenon of *ageing* (i.e. the strong dependence of the dynamical correlation functions over the waiting time). Our model is the so-called *random random walk*, the toy model of a directed polymer evolving in a random medium.

Ageing is a very interesting phenomenon that has been observed in spin glasses [1–4], but is likely to be present in many other materials (for example rubber), and to be a crucial signature of the behaviour of a strongly disordered system. In brief we can say we have ageing if the response of the system to a perturbation strongly depends on the time t_w (waiting time) during which the system has been kept in the low-temperature phase before starting the measurement.

More precisely in an ageing experiment one brings the system to the low temperature phase at time $t_0=0$. One leaves the system quiescent up to the time t_w, and at t_w add a small perturbation (small enough that the linear response theory can be applied). One eventually measures the response function $R(t, t_w)$ at time $t+t_w$. Alternatively one can measure the correlation function $C(t, t_w)$ between two quantities measured at time t_w and at time $t+t_w$.

In the region where the observation time it is very much smaller than the waiting time, $t \ll t_w$, R and C are independent of t_w and they are related by the fluctuation–dissipation theorem. Ageing is present if R and C are strongly dependent on t_w in the region where t is of the order of t_w. The phenomenon of ageing has been observed not only in real spin glasses, but also in numerical simulations of short-range [5] and long-range [6] spin glasses. In this last case the mean field approach [7, 8] is exact, so that an analytic study of ageing should be possible.

It seems natural that in the region of large times a simple scaling behaviour should hold. Different suggestions have been put forward. One simple possibility [9] is that in the region where both t and t_w are large

$$C(t, t_w) = t^{-\mu} f\left(\frac{t}{t_w}\right) \tag{1}$$

where the exponent μ may be equal to zero. Unfortunately, at the moment we do not have complete justification of this formula.

We can sketch a simple qualitative explanation. We suppose that the system is described by a potential with a corrugated landscape, where many barriers of all possible heights are present. Such a system (considered in the thermodynamic limit) is always out of equilibrium. Its dynamics are given by the never-ending search of the absolute minimum by crossing potential barriers that can be arbitrarily large.

Let us assume that the available phase space does not increase too much with the height of the barriers that have been crossed. In this case we can suppose that at time t_w and temperature T the system has explored all the phase space that can be reached from the origin by crossing barriers that are smaller than $T\ln(t_w)$. The system therefore will remain near the bottom of the explored phase space up to the moment at which it crosses one other large barrier. This can happen only at times of the order of t_w. If we assume that the shape of the deepest minimum found does not depend on the size of the explored region we are led to (1). In this way we can explain the dependence on t/t_w. One would need a more careful analysis to derive the value of μ, which may depend on the detailed quantity whose correlation function is computed. It should be noted that if

$$\lim_{t\to\infty}\lim_{t_W\to\infty} C(t,t_w)=\text{const}\neq 0 \tag{2}$$

(i.e. if C goes to a non-zero finite limit in the region where both t and t_w are large and $t \ll t_w$) then $\mu=0$.

The aim of this note is to discuss the behaviour of a very simple system, the one-dimensional random walk in a random environment [11]. In this simple case many analytic results are available (for a review see [10]). Our numerical simulations will show that the ageing phenomenon is present also in such a simple setting.

The model we consider is a particle whose equilibrium probability distribution is given by

$$P(x)\propto \mathrm{e}^{-\beta(V(x)+\lambda x^2)} \tag{3}$$

where V is a random Gaussian quantity with zero average and correlations

$$\overline{(V(x)-V(y))^2}=|x-y|^{\alpha}. \tag{4}$$

In the case $\alpha=1$ (the one that we will mainly consider in the following) the force $F(x)\equiv-\mathrm{d}V/\mathrm{d}x$ is uncorrelated from point to point. One introduces the contribution λx^2 to regularize the static equilibrium distribution, and $\lambda\to 0$ in the relevant limit. At equilibirium this model coincides with the toy model that was introduced in the study of the behaviour of interfaces in random media (see [12] and references therein).

It is known that [13]

$$\begin{aligned}&\overline{\langle x\rangle^2}\sim\lambda^{-4/3}\\ &\overline{\langle x^2\rangle-\langle x\rangle^2}\sim\lambda^{-1}\\ &\overline{\ln(\langle x^2\rangle-\langle x\rangle^2)}\sim O(1).\end{aligned} \tag{5}$$

The first equation implies that for non-zero λ the particle is contained inside a region of radius of order $\lambda^{-2/3}$. In the $\lambda\to 0$ limit the particle is always at infinity at equilibrium (the potential V is unbound).

In most cases the particle is localized around the minimum. The most probable value of $\langle x^2\rangle-\langle x\rangle^2$ is of order one, although rare events, in which the potential has two widely separated nearly degenerate minima, do contribute strongly to the average of $\langle x^2\rangle-\langle x\rangle^2$, making it of order λ^{-1}.

The dynamics of this model is a random walk biased to produce the equilibrium distribution (3) at large times. More precisely the model can be defined by introducing a one-dimensional lattice. If the particle is at point x at time t (x and t taking integer values) then p_x is the probability that at time $t+1$ the particle is at site $x+1$, and $1-p_x$ the probability that it is at site $x-1$. p_x is a random variable with zero average (typically uniformly distributed in $(0, 1)$, not including the interval limits).

It is easy to see that the equilibrium potential corresponding to this dynamics satisfy the condition

$$V(x+1)-V(x)=-\ln\left(\frac{p_x}{1-p_{x+1}}\right). \tag{6}$$

Consequently at large distance the difference of the potentials $V(x)-V(y)$ is approximately Gaussian, with $\alpha=1$.

This dynamics has been widely investigated [10]. It can be proven [14] that for the particle to go from x to y it has to cross (with high probability) a barrier that is of order $|x-y|^{1/2}$. If the system is in $x=0$ at $t=0$, at a large time t it will be in $x\sim\ln(t)^2$.

In the following we will discuss the phenomenon of ageing in this model. The simplest quantity one could study is the correlation function

$$C(t, t_w)=\overline{(x(t+t_w)-x(t_w))^2} \tag{7}$$

that for large t (in the asymptotic region studied by Sinai, where $t\gg t_w$) behaves as $\ln(t)^4$. But this is not a good choice. In the region of large times, for $t\ll t_w$, this function behaves as $\ln(t)^3$. The reasons for this behaviour are quite clear. If we assume that the probability distribution for the dynamics at time t may be mimicked by the probability distribution for the statics, where λ is chosen so that the value of $\langle x\rangle$ coincides with the correct one, we find from the previous equations that $\langle x^2\rangle-\langle x\rangle^2$ is of order $\ln(t)^3$. In other words the system has explored a region of size $\ln(t)^2$ and the probability of having two nearly equal minima inside this region vanishes as $1/\ln(t)$ as follows from the analysis of [13].

The previous analysis implies that the ageing properties of the function C are not simple and the scaling law in (1) cannot hold with $\mu=0$. So we found convenient to consider the following correlation.

$$L(t, t_w)\equiv\overline{\ln((x(t+t_w)-x(t_w))^2)} \tag{8}$$

or equivalently $M(t, t_w)\equiv\exp(L(t, t_w))$. M is not far from the most likely value of $(x(t+t_w)-x(t_w))^2$. In this case we expect that in the region of large times, when $t\ll t_w$:

$$M(t, t_w)=\text{constant}+O(\ln(t)^{-1}). \tag{9}$$

All that said, it seemed natural to set up a numerical simulation to verify the validity of the scaling law:

$$M(t, t_w)=f\left(\frac{t}{t_w}\right)+O(\ln(t)^{-1}). \tag{10}$$

We have generated 15×10^3 realizations of the one-dimensional random potential. For each of these realizations of the random potential we have observed the random walker travelling for 2^K steps and we have measured the correlation $M(t, t_w)$ at times t and t_w equal to 2^k, with $k=1, \ldots, K-1$. The spatial lattice had an infinite extent, i.e. the particle would never hit a boundary. We have taken $K=21$, i.e. performed order of 2×10^6 sweeps per sample.

We show in figure 1, $\langle \ln(|x(t)|\rangle$ versus $\ln(\ln(t+1))$ for different waiting time t_w, scaled logarithmically. In figure 1 we have drawn continuous lines joining the points of equal t_w. Higher curves correspond to lower t_w. The first continuous line from the top joins points taken after no waiting ($t_w=0$), the second line is for $t_w=1$, the third for $t_w=2$, the fourth for $t_w=4$, and so on, with an exponential progression.

The upper lines of figure 1 have $t \gg t_w$, and in the right part of the plot they scale with good precision as expected. Such straight lines have a slope very close to 2.

In figure 2 we plot the same points, but we join points of constant t/t_w ratio. Here the t/t_w scaling is quite clear. Indeed the lines at intermediate values of t/t_w tend for large t to a constant value. The distance reached from the walker at time t depends only on the ratio t/t_w. If we double the observation time t but we also double the waiting time t_w the average distance covered by the walker does not change. Figures 1 and 2, and the other figures we will show, allow us to draw the main conclusion of this note.

> For large t and t_w, t of order t_w, the correlation functions behave as a universal function of t/t_w.

The simplicity of the 1-dimensional (toy) model allows us to obtain very compelling numerical evidence for such an effect.

In figure 3 we plot again the same data, but this time we select results with both t and t_w larger than 32, and we plot $\exp(\langle \ln(|x(t)|)\rangle)$ versus $\ln((t+1)/(t_w+1))$. The data fall with good precision on a single scaling curve. In the left part of the x-axis (small t compared with t_w) the universal curve is a constant (aside from small corrections), while in the large t/t_w region it behaves as $\ln(t/t_w)^2$ (according to the Sinai [14] scaling law).

Similar results can be obtained considering energy–energy correlation defined by

$$C_E(t, t_w) \equiv \overline{(V(x(t)) - V(x(t_w)))^2} \tag{11}$$

where V is defined by equation (6).

We expect that $C_E(t, t_w) \sim \ln(t/t_w)^2$, independent of the value of α. For the same reason we also expect that the expectation value of V:

$$E(t) = \overline{(V(x(t))} \tag{12}$$

behaves as $-\ln(t)$ for large t, independent of α.

Such a scaling law is very well satisfied. Figures 4–6 are analogous to figures 1–3, and show the analogous scaling laws (which in this case are even better). In figure 4 we give $-V(t)$ as a function of $\ln(t)$, and we join points of equal t_w. The large t linear behaviour is very clear. In figure 5 we join points of equal t/t_w ratio. Again, the curves tend to constant values. In figure 6 we show the rescaling of the points with $t>32$, $t_w>32$, which again show very good accuracy. Here it is very clear, for example, that in the $t<t_w$ asymptotic region the walker does not gain any energy.

The numerical results we have presented here bring good evidence for the correctness of the simple ageing scaling relation

$$\mathscr{C}(t, t_w) \simeq \mathscr{F}\left(\frac{t}{t_w}\right) \tag{13}$$

for the correlation function of the logarithm of the distance and of the energy.

In our model simple ageing is correct. This conclusion seems to us interesting because of its potential general implications, and because the simplicity of the model

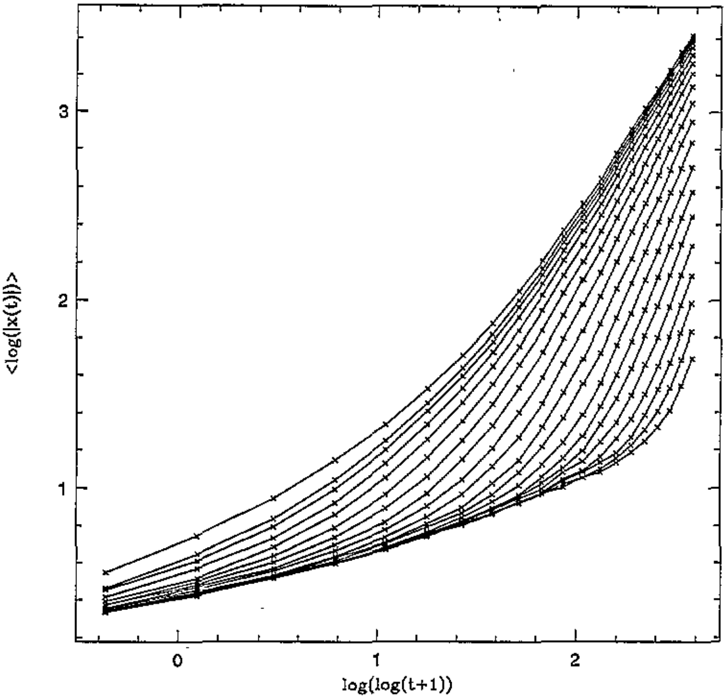

Figure 1. The average logarithm of the distance reached by the walker after t steps as a function of $\ln(\ln(t))$. Here the lines join points with constant t_w. Lines from top to bottom are from lower to higher t_w.

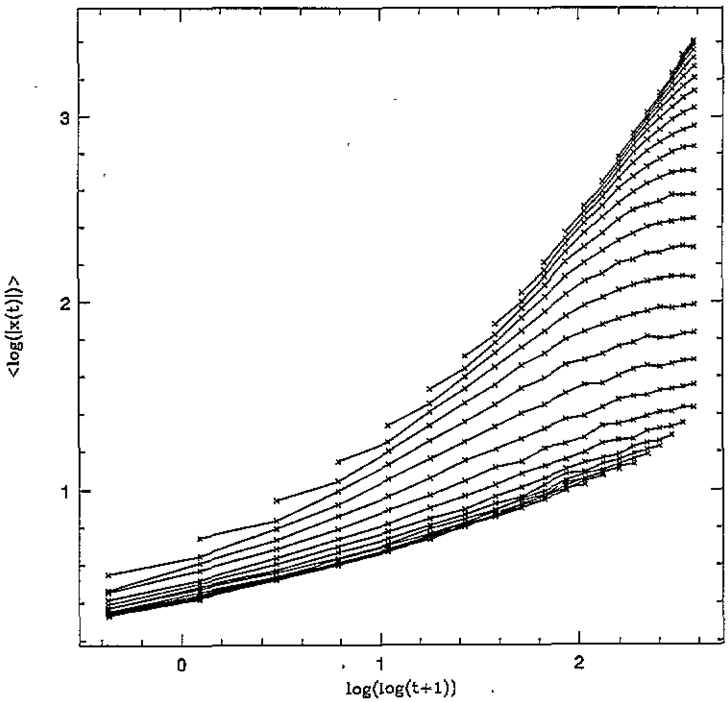

Figure 2. Same points as figure 1, but here the lines join points of constant t/t_w.

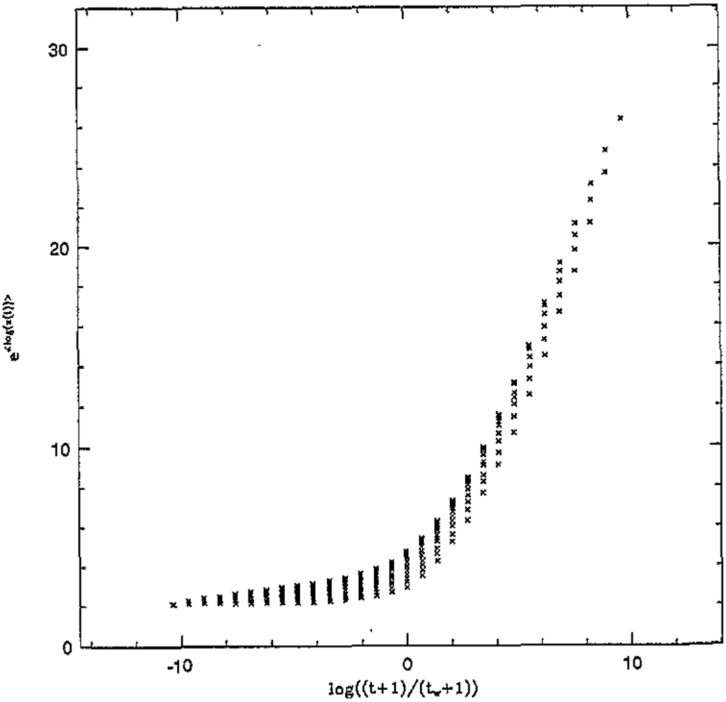

Figure 3. A selection of the points from figures 1 and 2, for large t and t_w, versus $\ln(t/t_w)$.

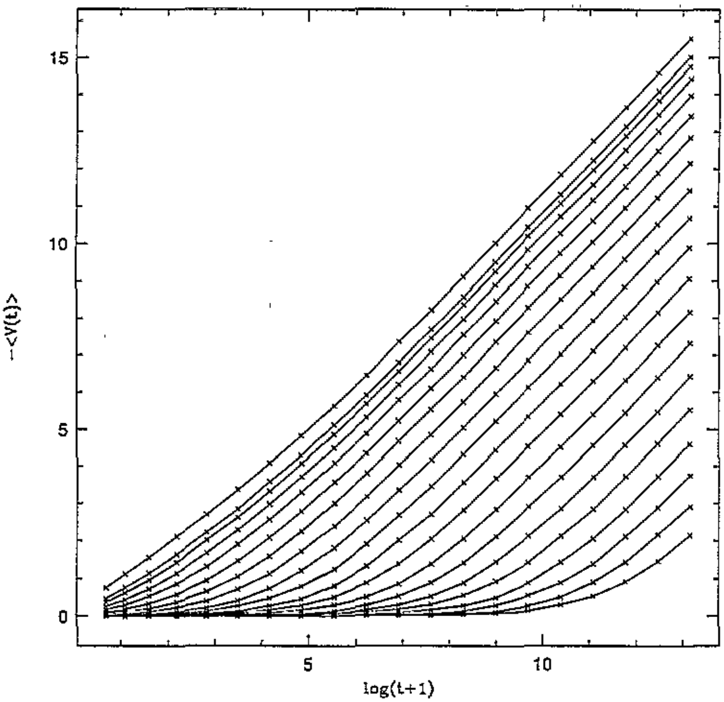

Figure 4. As in figure 1 (lines join points with constant t_w), but for $-V(t)$ versus $\ln(t)$.

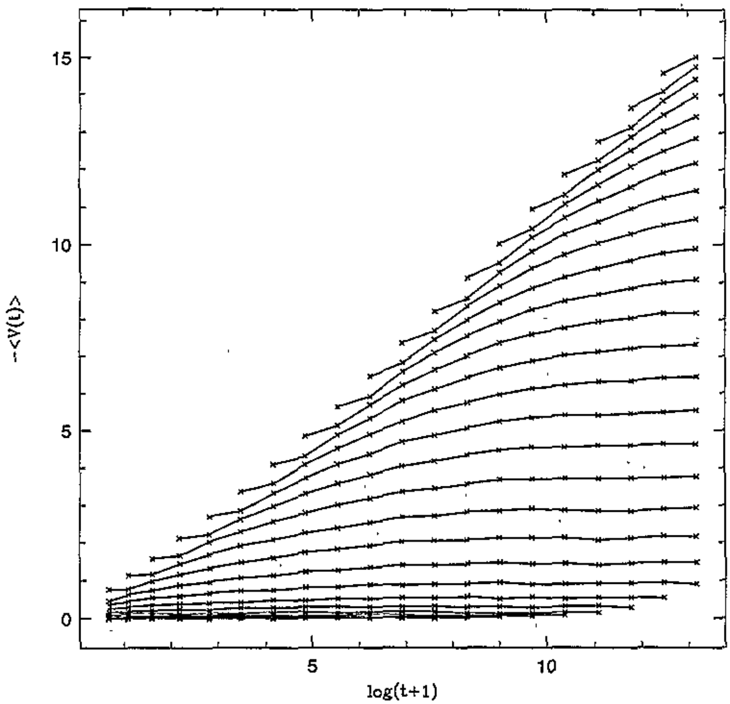

Figure 5. As in figure 4, but lines join points of constant t/t_w.

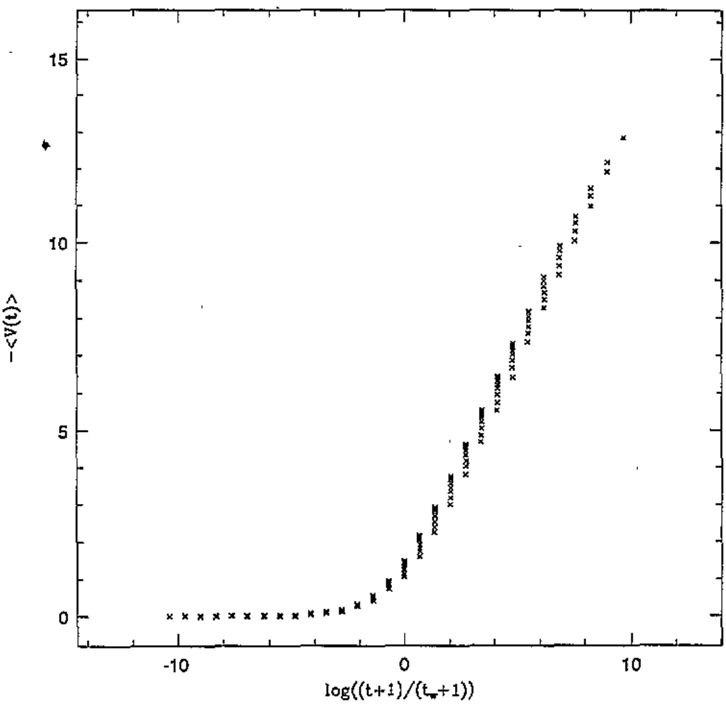

Figure 6. As in figure 3, but for $-V(t)$.

is such that a more sound analytic (and maybe rigorous) derivation of these results does not seem impossible.

As a further development of this work, we notice that it could be interesting to study finite volume effects for large, but finite lattice size (i.e. with reflecting or periodic boundary conditions) and to study the modification of our results for very large time. The extension of the model to higher-dimensional cases, where the structure of minima is more complex, will probably be instructive.

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